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Symmetric Kinematic Transformation Pair Using Euler Parameters

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THIS Note derives a kinematic transformation using Euler parameters (quaternion elements), which results in a symmetric pair of direct and inverse transformations. The construction of direction cosine matrices is not required, nor do the formulations depend upon quaternion algebra. While quaternion transformations offer several computational advantages[1,2] over direction cosines, the rather obscure algebra of hyperimaginary numbers hinders the analytic treatment of individual vector components.

In the method described here, a three-vector whose components are to be expressed in either of two orthogonal basis sets will be replaced by a four-vector in order to achieve the simple symmetry in the transformations. Both Kane[3] and Morton[4] have shown the advantages of using four-vector representations of certain quantities in dynamics. The transformation presented here facilitates these methods of analysis by eliminating the need to revert temporarily to three-vectors during coordinate transformation.

If A and B are two coordinate systems defined by dextral, orthonormal triads $(\boldsymbol{a}_1,\boldsymbol{a}_2,\boldsymbol{a}_3)$ and $(\boldsymbol{b}_1,\boldsymbol{b}_2,\boldsymbol{b}_3)$ respectively, and if the orientation of B relative to A is such that a rotation of A in the right-hand sense about the direction defined by the unit vector λ by an amount θ causes the $\boldsymbol{a}_j$ vectors to become parallel to the respective $\boldsymbol{b}_i$ vectors, then the orientation of B relative to A can be expressed by the use of the direction cosine matrix (DCM) C as

$$\boldsymbol{b} = C\boldsymbol{a} \tag{1}$$

where $\boldsymbol{a} = [\boldsymbol{a}_1,\boldsymbol{a}_2,\boldsymbol{a}_3]^T$ and $\boldsymbol{b} = [\boldsymbol{b}_1,\boldsymbol{b}_2,\boldsymbol{b}_3]^T$ and the elements of C are computed from the Euler parameters

$$\epsilon_1 \triangleq \lambda_1 \sin\frac{\theta}{2} \tag{2a}$$

$$\epsilon_2 \triangleq \lambda_2 \sin\frac{\theta}{2} \tag{2b}$$

$$\epsilon_3 \triangleq \lambda_3 \sin\frac{\theta}{2} \tag{2c}$$

$$\epsilon_4 \triangleq \cos\frac{\theta}{2} \tag{2d}$$

where $\lambda_i = \lambda \cdot \boldsymbol{a}_i = \lambda \cdot \boldsymbol{b}_i$, with the result (c.f. Ref. 3) that

$$\boldsymbol{b}_1 = (\epsilon_1^2 - \epsilon_2^2 - \epsilon_3^2 + \epsilon_4^2)\boldsymbol{a}_1 + 2(\epsilon_1\epsilon_2 + \epsilon_3\epsilon_4)\boldsymbol{a}_2 + 2(\epsilon_3\epsilon_1 - \epsilon_4\epsilon_2)\boldsymbol{a}_3 \tag{3}$$

Using the normalization constraint on the Euler parameters

$$\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 + \epsilon_4^2 = 1 \tag{4}$$

to eliminate the negative terms ϵ_2^2 and ϵ_3^2 from the first part in Eq. (3) yields

$$\boldsymbol{b}_1 = -\boldsymbol{a}_1 + 2[(\epsilon_4^2 + \epsilon_1^2)\boldsymbol{a}_1 + (\epsilon_1\epsilon_2 + \epsilon_4\epsilon_3)\boldsymbol{a}_2 + (\epsilon_3\epsilon_1 - \epsilon_4\epsilon_2)\boldsymbol{a}_3] \tag{5}$$

Similarly,

$$\boldsymbol{b}_2 = -\boldsymbol{a}_2 + 2[(\epsilon_1\epsilon_2 - \epsilon_4\epsilon_3)\boldsymbol{a}_1 + (\epsilon_4^2 + \epsilon_2^2)\boldsymbol{a}_2 + (\epsilon_2\epsilon_3 + \epsilon_4\epsilon_1)\boldsymbol{a}_3] \tag{6}$$

$$\boldsymbol{b}_3 = -\boldsymbol{a}_3 + 2[(\epsilon_1\epsilon_3 + \epsilon_4\epsilon_2)\boldsymbol{a}_1 + (\epsilon_2\epsilon_3 - \epsilon_4\epsilon_1)\boldsymbol{a}_2 + (\epsilon_4^2 + \epsilon_3^2)\boldsymbol{a}_3] \tag{7}$$

In matrix form, Eqs. (5–7) become

$$\boldsymbol{b} = -\boldsymbol{a} + 2(E' + \epsilon_4\epsilon')\boldsymbol{a} = C\boldsymbol{a} \tag{8}$$

with

$$E' = \begin{bmatrix} \epsilon_1^2 & \epsilon_1\epsilon_2 & \epsilon_1\epsilon_3 \\ \epsilon_2\epsilon_1 & \epsilon_2^2 & \epsilon_2\epsilon_3 \\ \epsilon_3\epsilon_1 & \epsilon_3\epsilon_2 & \epsilon_3^2 \end{bmatrix} \tag{9}$$

$$\epsilon' = \begin{bmatrix} \epsilon_4 & \epsilon_3 & -\epsilon_2 \\ -\epsilon_3 & \epsilon_4 & \epsilon_1 \\ \epsilon_2 & \epsilon_1 & \epsilon_4 \end{bmatrix} \tag{10}$$

It should be noted that Eq. (8) is formally somewhat similar to Eqs. (12–13b) in Ref. 1 and that it contains the same information as Eq. (21) on p. 14 of Ref. 3; however, there is an important symmetry in Eq. (8) (obtained by eliminating the negative quadratic terms in the Euler parameters rather than the positive ones) that will be exploited later. By augmenting the matrices E' and ϵ' with an extra row and column, some additional symmetry is achieved. Define the 4×4 matrices

$$E \triangleq \beta\beta^T \tag{11}$$

where

$$\beta = [\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4]^T \tag{12}$$

$$R \triangleq \begin{bmatrix} \epsilon_4 & \epsilon_3 & -\epsilon_2 & \epsilon_1 \\ -\epsilon_3 & \epsilon_4 & \epsilon_1 & \epsilon_2 \\ \epsilon_2 & -\epsilon_1 & \epsilon_4 & \epsilon_3 \\ -\epsilon_1 & -\epsilon_2 & -\epsilon_3 & \epsilon_4 \end{bmatrix} \tag{13}$$

Letting

$$\boldsymbol{a}^* = [\boldsymbol{a}_1,\boldsymbol{a}_2,\boldsymbol{a}_3,0]^T \tag{14a}$$

$$\boldsymbol{b}^* = [\boldsymbol{b}_1,\boldsymbol{b}_2,\boldsymbol{b}_3,0]^T \tag{14b}$$

gives

$$\boldsymbol{b}^* = -\boldsymbol{a}^* + 2(E + \epsilon_4 R)\boldsymbol{a}^* = C^*\boldsymbol{a}^* \tag{15}$$

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with

$$C^* \triangleq -I + 2(E + \epsilon_4 R) \tag{16}$$

The inverse of C^* is obtained via Eq. (8), where

$$C = -I + 2(E' + \epsilon_4 \epsilon') \tag{17}$$

Since the transformation in Eq. (8) is orthogonal,

$$C^{-1} = C^T = -I + 2(E' + \epsilon_4 \epsilon'^T) \tag{18}$$

Again, by augmenting ϵ'^T with a fourth row and column, and defining

$$Q \triangleq \begin{bmatrix} \epsilon_4 & -\epsilon_3 & \epsilon_2 & \epsilon_1 \\ \epsilon_3 & \epsilon_4 & -\epsilon_1 & \epsilon_2 \\ -\epsilon_2 & \epsilon_1 & \epsilon_4 & \epsilon_3 \\ -\epsilon_1 & -\epsilon_2 & -\epsilon_3 & \epsilon_4 \end{bmatrix} \tag{19}$$

yields

$$a^* = -b^* + 2(E + \epsilon_4 Q) b^* = C^{*-1} b^* \tag{20}$$

with

$$C^{*-1} \triangleq -I + 2(E + \epsilon_4 Q) \tag{21}$$

Combining Eqs. (16) and (21) yields the relation

$$C^{*-1} = C^* + 4\epsilon_4 \tilde{\epsilon} \tag{22}$$

where $\tilde{\epsilon}$ is the skew-symmetric matrix defined by

$$\tilde{\epsilon} \triangleq \begin{bmatrix} 0 & -\epsilon_3 & \epsilon_2 & 0 \\ \epsilon_3 & 0 & -\epsilon_1 & 0 \\ -\epsilon_2 & \epsilon_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{23}$$

In summary, this note has developed a kinematic transformation using Euler parameters directly and without the necessity of constructing a direction cosine matrix or using quaternion algebra. The transformation pair given by Eqs. (16) and (21) possesses an interesting symmetry and allows for the kinematic transformation of dynamic quantities expressed as four-vectors without having to change their representation back to the three-vectors form.

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Model Reference Adaptive Control for Large Structural Systems

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Introduction

A MODEL reference adaptive control technique as suggested by Sobel et al.[1] has recently been applied to the control of large structural systems.[2,3] In this technique, a reference model incorporating the desirable properties for the plant is selected and, by using command generator tracker procedure[4] and Liapunov stability theory, a control is designed that makes the error vector between the model and plant outputs approach zero asymptotically. The control law is chosen as a linear combination of output errors, reference model states, and reference model inputs, with the adaptive gains being the sum of the integral and proportional gains. Using this procedure, Bar-Kana and co-workers[2] have shown that when collocated actuators and combined position and rate sensors are used, the output error approaches zero, provided the ratio of the position-to-rate output is limited by a function dependent on the damping ratio and the lowest structural frequency.

In this Note, we improve on this limit and show, by constructing a suitable Liapunov function, that the ratio of position-to-rate output is limited by twice the product of the damping ratio and the lowest structural frequency. As well, we show that other control laws (e.g., the relay type) can be designed.

Dynamics of Structures

By using the finite-element method, the dynamics of a linear structure can be represented by the system of differential equations as

$$M\ddot{q} + Kq = \bar{B}u \tag{1}$$

$$y = \bar{C}(\alpha q + \dot{q}) \tag{2}$$

where M is an $(n \times n)$ mass matrix, q an n-vector of coordinates, K an $(n \times n)$ stiffness matrix, $\bar{B}$ an $(n \times m)$ control influence matrix, u and y m-dimensional input and output vectors, respectively, $\bar{C}$ an $(m \times n)$ measurement distribution matrix, and α the weighting factor of position-to-rate measurement.

Let Φ denote the modal matrix, such that

$$\Phi^T M \Phi = I_n = (n \times n) \text{ unit matrix}$$

$$\Phi^T K \Phi = \text{diag}(\omega_1^2, \ldots, \omega_i^2, \ldots, \omega_n^2) \equiv \Omega^2$$

where T denotes the transpose of a matrix. In terms of the modal coordinates $\eta = \Phi^{-1} q$, Eqs. (1) and (2) can be written as

$$\ddot{\eta} + \Omega^2 \eta = \Phi^T \bar{B} u \tag{3}$$

$$y = \bar{C}\Phi(\alpha\eta + \dot{\eta}) \tag{4}$$

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